Time dependence of joint entropy of oscillating quantum systems

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Abstract

The time dependent entropy (or Leipnik's entropy) of harmonic and damped harmonic oscillators is extensively investigated by using time dependent wave function obtained by the Feynman path integral method. Our results for simple harmonic oscillator are in agrement with the literature. However, the joint entropy of damped harmonic oscillator shows remarkable discontinuity with time for certain values of damping factor. According to the results, the envelop of the joint entropy curve increases with time monotonically. This results is the general properties of the envelop of the joint entropy curve for quantum systems.

Keywords: Path integral, joint entropy, simple harmonic oscillator, damped harmonic oscillator, negative joint entropy

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I. INTRODUCTION

The investigation of time dependent entropy of the quantum mechanical systems attracts much attention in recent years. For both open and closed quantum systems, the different information-theoretic entropy measures have been discussed [2, 3, 4]. In contrast, the joint entropy [5, 6] can also be used to measure the loss of information, related to evolving pure quantum states [7]. The joint entropy of the physical systems which are named MACS (maximal classical states) were conjectured by Dunkel and Trigger [8]. According to Ref. [8], the joint entropy of the quantum mechanical systems increase monotonically with time but this results are not sufficient for simple harmonic oscillator [9].

The aim of this study is to calculate the complete joint entropy information analytically for simple harmonic and damped harmonic oscillator systems.

This paper is organized as follows. In section II, we explain fundamental definitions needed for the calculations. In section III, we deal with calculation and results for harmonic oscillator systems. Moreover, we obtain the analytical solution of Kernel, wave function in both coordinate and momentum space and its joint entropy. We also obtain same quantities for damped harmonic oscillator case. Finally, we present the conclusion in section IV.

II. FUNDAMENTAL DEFINITIONS

We deal with a classical system with $d = sN$ degrees of freedom, where N is the particle number and s is number of spatial dimensions [8]. We assume that the density function $g(x, p, t) = g(x_1, ..., x_d, p_1, ..., p_d, t)$ which is the non-negative time dependent phase space density function of the system has been normalized to unity,

$$\int dxdpg(x, p, t) = 1. \tag{1}$$

The Gibbs-Shannon entropy is described by

$$S(t) = -\frac{1}{N!}\int dxdpg(x, p, t)ln(h^d g(x, p, t)), \tag{2}$$

where $h = 2\pi\hbar$ is the Planck constant. Schrödinger wave equation with the Born interpretation [10] is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi. \tag{3}$$

The quantum probability densities are defined in position and momentum spaces as $|\psi(x,t)|^2$ and $|\tilde{\psi}(p,t)|^2$, where $|\tilde{\psi}(p,t)|^2$ is given as

$$\tilde{\psi}(p,t) = \int \frac{dx e^{-ipx/\hbar}}{(2\pi\hbar)^{d/2}} \psi(x,t). \tag{4}$$

Leipnik proposed the product function as [8]

$$g_j(x,p,t) = |\psi(x,t)|^2 |\tilde{\psi}(p,t)|^2 \geq 0. \tag{5}$$

Substituting Eq. (5) into Eq. (2), we get the joint entropy $S_j(t)$ for the pure state $\psi(x,t)$ or equivalently it can be written in the following form [8]

$$\begin{aligned} S_j(t) \; &= \; -\int dx |\psi(x,t)|^2 \ln|\psi(x,t)|^2 - \int dp |\tilde{\psi}(p,t)|^2 \ln|\tilde{\psi}(p,t)|^2 - \\ &- \; \ln h^d. \end{aligned} \tag{6}$$

We find time dependent wave function by means of the Feynman path integral which has form [11]

$$\begin{aligned} K(x'',t'';x',t') \; &= \; \int_{x'=x(t')}^{x''=x(t'')} Dx(t) e^{\frac{i}{\hbar} S[x(t)]} \\ &= \; \int_{x'}^{x''} Dx(t) e^{\frac{i}{\hbar} \int_{t'}^{t''} L[x,\dot{x},t] dt}. \end{aligned} \tag{7}$$

The Feynman kernel can be related to the time dependent Schrödinger's wave function

$$K(x'',t'';x',t') = \sum_{n=0}^{\infty} \psi_n^*(x',t') \psi_n(x'',t''). \tag{8}$$

The propagator in semiclassical approximation reads

$$K(x'',t'';x',t') = \Big[\frac{i}{2\pi\hbar} \frac{\partial^2}{\partial x' \partial x''} S_{cl}(x'',t'';x',t')\Big]^{1/2} e^{\frac{i}{\hbar} S_{cl}(x'',t'';x',t')}. \tag{9}$$

The prefactor is often referred to as the Van Vleck-Pauli-Morette determinant [12, 13]. The $F(x'',t'';x',t')$ is given by

$$F(x'',t'';x',t') = \Big[\frac{i}{2\pi\hbar} \frac{\partial^2}{\partial x' \partial x''} S_{cl}(x'',t'';x',t')\Big]^{1/2}. \tag{10}$$

III. CALCULATION AND RESULTS

A. Simple Harmonic Oscillator (SHO)

To get the path integral solution for the SHO, we must calculate its action function. The Lagrangian of the system is given by

$$L(x, \dot{x}, t) = \frac{m}{2}(\dot{x}^2 - \frac{1}{2}\omega^2 x^2) \tag{11}$$

Following a straightforward calculation, it is given by:

$$S(x_{cl}(t''), x_{cl}(t')) = \frac{m\omega}{2\sin\omega t}[(x''^2_{cl} + x'^2_{cl})\cos\omega t - 2x'_{cl}x''_{cl}] \tag{12}$$

with $t = t'' - t'$ and $x'_{cl} = x_0, x''_{cl} = x$. Substituting Eq. (9) into Eq. (7), we obtain the Feynman kernel [11]:

$$K(x, x_0; t) = (\frac{m\omega}{2\pi\hbar i\sin\omega t})^{\frac{1}{2}} \exp\{-\frac{m\omega}{2i\hbar}[(x^2 + {x_0}^2)\cot\omega t - \frac{2x_0 x}{\sin\omega t}]\}. \tag{13}$$

By the use of the Mehler-formula

$$e^{-(x^2+y^2)/2}\sum_{n=0}^{\infty}\frac{1}{n!}(\frac{z}{2})^2 H_n(x)H_n(y) \;=\; \frac{1}{\sqrt{1-z^2}}\exp[\frac{4xyz - (x^2+y^2)(1+z^2)}{2(1-z^2)}] \tag{14}$$

where H_n is Hermite polynomials, we can write the Feynman kernel defining $x \equiv \sqrt{m\omega/\hbar}x_0$, $y \equiv \sqrt{m\omega/\hbar}x$ and $z = e^{-i\omega T}$

$$K(x, x_0; t) = \sum_{n=0}^{\infty} e^{-itE_n/\hbar}\Psi^*(x_0)\Psi(x) \tag{15}$$

with energy-spectrum and wave-functions:

$$E_n = \hbar\omega(n + \frac{1}{2}), \tag{16}$$

$$\Psi_n(x) = (\frac{m\omega}{2^{2n}\pi\hbar n!^2})^{\frac{1}{4}} H_n(\sqrt{\frac{m\omega}{\hbar}}x)\exp(-\frac{m\omega}{2\hbar}x^2). \tag{17}$$

Time dependent wave function of the SHO is defined as

$$\Psi(x, t) = \int K(x, x_0; t)\Psi(x_0, 0)dx_0. \tag{18}$$

It can be written as

$$\Psi(x, t) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}\exp\left\{-\frac{\bar{\alpha}}{4} - \frac{\alpha^2}{2} - \frac{i\omega t}{2}\right\}\exp\left[-\frac{\bar{\alpha}^2}{4}e^{-2i\omega t} + \alpha\bar{\alpha}e^{-i\omega t}\right] \tag{19}$$

where $\bar{x}$ or $\bar{\alpha}$ is mean of the Gaussian curve. The probability density has

$$|\Psi(x,t)|^2 = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/2} \exp\Big[-(\alpha - \bar{\alpha}\cos\omega t)^2\Big] \tag{20}$$

where $\alpha = \sqrt{\frac{m\omega}{\hbar}}x$. Thus it can be written as

$$|\Psi(x,t)|^2 = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/2} \exp\Big[-\frac{m\omega}{\hbar}(x - \bar{x}\cos\omega t)^2\Big] \tag{21}$$

This has been shown in Fig.1. In momentum space, the probability density has the form

$$|\tilde{\psi}(p,t)|^2 = \Big(\frac{1}{m\omega\pi\hbar}\Big)^{1/2} \exp\Big[\frac{-p^2}{m\omega\hbar} + \frac{m\omega\bar{x}^2}{2\hbar}\Big(\cos 2\omega(t) - 1\Big) - \frac{2p\bar{x}}{\hbar}\sin\omega(t)\Big]. \tag{22}$$

The joint entropy of harmonic oscillator becomes

$$S_j(t) = \ln\frac{e}{2} + \frac{4m\omega}{\hbar}\bar{x}^2\sin^2\omega(t). \tag{23}$$

In Fig.2, the joint entropy of this system was plotted by using Mathematica in three dimension. As known from fundamental quantum mechanics and classical dynamics, displacement of simple harmonic oscillator from equilibrium depends on harmonic functions (e.g sine or cosine function). Therefore, other properties of the SHO systems indicate the same harmonic behavior. If the frequency of the SHO is sufficiently small, the system shows the same behavior as the free particle[8]. As seen from Fig.3 and Fig.4, envelop of the sinusoidal curve is also monotonically increase with omega and constant with time at constant omega, respectively. When the frequency increases, the joint entropy of this system indicates a fluctuation with increasing amplitude with time. If t goes to zero, it is important that Eq.(20) is in agreement with following general inequality for the joint entropy:

$$S_j(t) \geq \ln(\frac{e}{2}) \tag{24}$$

originally derived by Leipnik for arbitrary one-dimensional one-particle wave functions.

B. Damped Harmonic Oscillator (DHO)

The DHO is very important physical system in all physical systems defining an interaction with its environment. The Lagrangian of the DHO is given by

$$L(x,\dot{x},t) = e^{\gamma t}\Big(\frac{m}{2}\dot{x}^2 - \frac{m}{2}\omega^2x^2 + j(t)x\Big). \tag{25}$$

Damped free particle kernel is

$$K(x,t;x_0,0)=\Big(\frac{\gamma m e^{\gamma t/2}}{4\pi i\hbar\sinh\frac{1}{2}\gamma t}\Big)^2\exp\Big(\frac{i\gamma m e^{\gamma t/2}}{4\hbar\sinh\frac{1}{2}\gamma t}(x-x_0)^2\Big). \tag{26}$$

The DHO kernel has the form [14]

$$K(x,t;x_0,0)=\Big(\frac{m\omega e^{\gamma t/2}}{2\pi i\hbar\sinh\omega t}\Big)^{1/2}\exp\Big(\frac{i}{\hbar}S_{cl}(x,x_0,t)\Big), \tag{27}$$

or explicitly

$$K(x,t;x_0,0)=\Big(\frac{m\omega e^{\gamma t/2}}{2\pi i\hbar\sin\omega t}\Big)^{1/2}\exp\Big[\frac{im}{2\hbar}(ax^2+2bx_0^2+2xx_0c+2xd+2x_0e-f)\Big]. \tag{28}$$

Where the coefficients a, b, c, d, f are [14]

$$a=(-\frac{\gamma}{2}+\omega\cot\omega t)e^{\gamma t}, \tag{29}$$

$$b=(\frac{\gamma}{2}+\omega\cot\omega t), \tag{30}$$

$$c=(-\frac{\omega}{\sin\omega t}e^{\gamma t}), \tag{31}$$

$$d=\frac{e^{\gamma t}}{m\sin\omega t}\int_0^t j(t')e^{\gamma t'/2}\sin\omega t'dt', \tag{32}$$

$$e=\frac{1}{m\sin\omega t}\int_0^t j(t')e^{\gamma t'/2}\sin\omega(t-t')dt', \tag{33}$$

$$f=\frac{1}{m^2\omega}\int_0^t\int_0^{t'} j(t')j(s)e^{\gamma(s+t'/2)}\sin\omega(t-t')\sin\omega s ds dt'. \tag{34}$$

The wave function $\psi_n(x,0)$ and energy eigenvalues become

$$\psi_n(x,0)\ =\ N_0H_n(\alpha_0x)\exp\Big[-\frac{1}{2}\alpha_0x^2\Big] \tag{35}$$

and

$$E_n=\Big(n+\frac{1}{2}\Big)\hbar\omega_0 \tag{36}$$

where $H_n(x)$ is the Hermite polynomial of order n and the coefficients are

$$\alpha_0=(\frac{m\omega}{\hbar})^{1/2},N_0=\frac{\alpha^{1/2}}{(2^nn!\sqrt{\pi})^{1/2}}. \tag{37}$$

The time dependent wave function is obtained as [14]

$$
\begin{aligned}
\psi_n(x,t) &= \int_{-\infty}^{\infty} dx_0 K(x,t;x_0,0)\psi(x,0) \\
&= N\frac{1}{(2^n n!)^{1/2}} \exp\Big\{-i\Big[\Big(n+\frac{1}{2}\Big)\cot^{-1}\times \\
&\times \Big(\frac{\gamma}{2\omega}+\cot\omega t+f\Big)\Big]\Big\}\exp[-(Ax^2+ \\
&+ 2Bx)]H_n[D(x-E)]. \qquad (38)
\end{aligned}
$$

To simplify the evaluation, we set $j(t) = 0$. Such that kernel and wave function of the DHO [15] become

$$
\begin{aligned}
K(x,t;x_0,0) &= \Big(\frac{m\omega e^{\gamma t/2}}{2\pi i\hbar\sin\omega t}\Big)^{1/2}\exp\Big[\frac{im}{4\hbar}\Big(\gamma(x_0^2-e^{\gamma t}x^2)+\frac{2\omega}{\sin\omega t}\times \\
&\times [(x_0^2+x^2e^{\gamma t})\cos\omega t-2e^{\gamma t/2}xx_0]\Big)\Big] \qquad (39)
\end{aligned}
$$

where $\omega = (\omega_0^2-\gamma^2/4)^{1/2}$ and

$$
\psi_n(x,t) = \frac{N}{(2^n n!)^{1/2}}\exp\Big\{-i\Big[\Big(n+\frac{1}{2}\Big)\cot^{-1}\Big(\frac{\gamma}{2\omega}+\cot\omega t\Big)\Big]\Big\}H_n[Dx]\exp[-Ax^2]. \qquad (40)
$$

Where D, A and N are

$$
D(t) = \frac{\alpha e^{\gamma t/2}}{\eta(t)\sin\omega t}, \qquad (41)
$$

$$
\eta^2(t) = \frac{\gamma^2}{4\omega^2}+\frac{\gamma}{\omega}\cos\omega t+\csc^2\omega t, \qquad (42)
$$

$$
A(t) = \frac{m\omega}{2\hbar}e^{\gamma t}\Big[\frac{1}{\eta^2(t)\sin^2\omega t}+i\Big(\frac{\gamma}{2\omega}-\cot\omega t+\frac{\gamma/2\omega+\cot\omega t}{\eta^2\sin^2\omega t}\Big)\Big], \qquad (43)
$$

and

$$
N(t) = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4}\frac{\exp(\frac{\gamma t}{4})}{\eta(t)(\sin\omega t)^{1/2}}. \qquad (44)
$$

The ground state wave function is given by

$$
\psi_0(x,t) = N(t)\exp\Big\{-i\Big[\Big(\frac{1}{2}\Big)\cot^{-1}\Big(\frac{\gamma}{2\omega}+\cot\omega t\Big)\Big]\Big\}\exp[-A(t)x^2]. \qquad (45)
$$

So the probability distribution in coordinate space becomes

$$
|\psi_0(x,t)|^2 = N(t)^2\exp[-2A'(t)x^2] \qquad (46)
$$

where A' is defined by

$$
A'(t) = \frac{m\omega}{2\hbar}e^{\gamma t}\Big[\frac{1}{\eta^2(t)\sin^2\omega t}\Big]. \qquad (47)
$$

The probability density in coordinate space is shown in Fig.5 and Fig.6 for the different values of γ. The probability density in momentum space can be written easily

$$|\psi_0(p,t)|^2 \;=\; \frac{N(t)^2}{\sqrt{2A(t)A(t)^\dagger\hbar}}\exp\Big[-\frac{p^2}{2\hbar^2}\frac{A'(t)}{A(t)A(t)^\dagger}\Big]. \tag{48}$$

The time dependent joint entropy can be obtained from Eq. (2) as

$$S_j(t) = N(t)^2\sqrt{\frac{\pi}{2A'(t)}}\Big[(\ln N(t)^2-\frac{1}{2})-\frac{1}{2}\sqrt{\frac{1}{2A(t)A(t)^\dagger}}\Big(\ln\frac{N(t)^2}{2A(t)A(t)^\dagger}-\frac{1}{2}\Big)\Big]-\ln 2\pi. \tag{49}$$

The joint entropy depends on damping factor γ. When $\gamma \to 0$, all the above results are converged to simple harmonic oscillator. However, when the $\gamma \neq 0$, the joint entropy has remarkably different features of the SHO. As can be seen in Fig.7 and Fig.8, the joint entropy of the DHO has very interesting properties. One of the most important properties of the joint entropy is the probability of taking values for small γ values. As we know from literature the joint entropy must be positive and monotonically increase.However, this system has different properties from literature because of periodically discontinuity of the joint entropy. On the other hand, envelop of this curve is also monotonically increase with time for large γ. As can be shown these results, the envelop of the joint entropy curves has general properties as monotonically increase for quantum systems. Thus, we have found that the joint entropy is depend on properties of investigated system.

IV. CONCLUSION

We have investigated the joint entropy for explicit time dependent solution of one-dimensional harmonic oscillators. We have obtained the time dependent wave function by means of Feynmann Path integral technique. Our results show that in the simple harmonic oscillator case, the joint entropy fluctuated with time and frequency. This result indicates that the information periodically transfer between harmonic oscillators.

On the other hand, in the DHO case, the joint entropy shows a remarkable smooth discontinuities with time. It also depends on choice of initial values of parameter i.e. ω. These results can be explained as the information exchange between harmonic oscillator and system which is supplied damping. But the information exchange appears in certain values of time for damping. If the damping factor increases, the information entropy has not periodicity anymore. Moreover, for certain values of the damping factor, the transfer of information between systems is exhausted.

V. ACKNOWLEDGEMENTS

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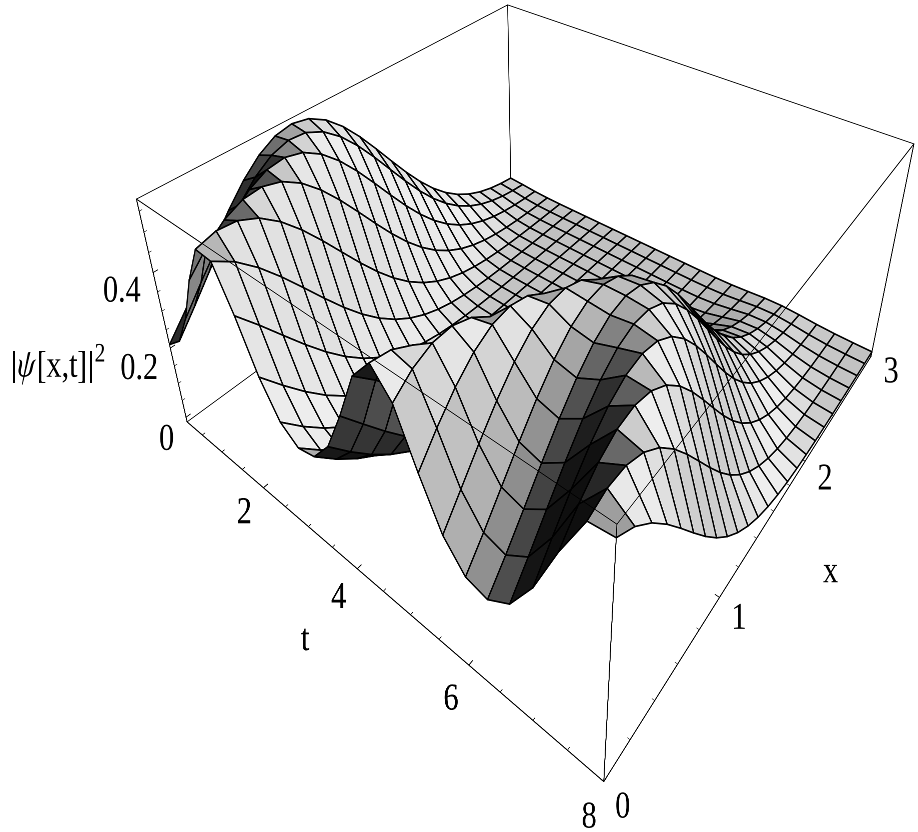

FIG. 1: $|\Psi(x,t)|^2$ versus time and coordinate

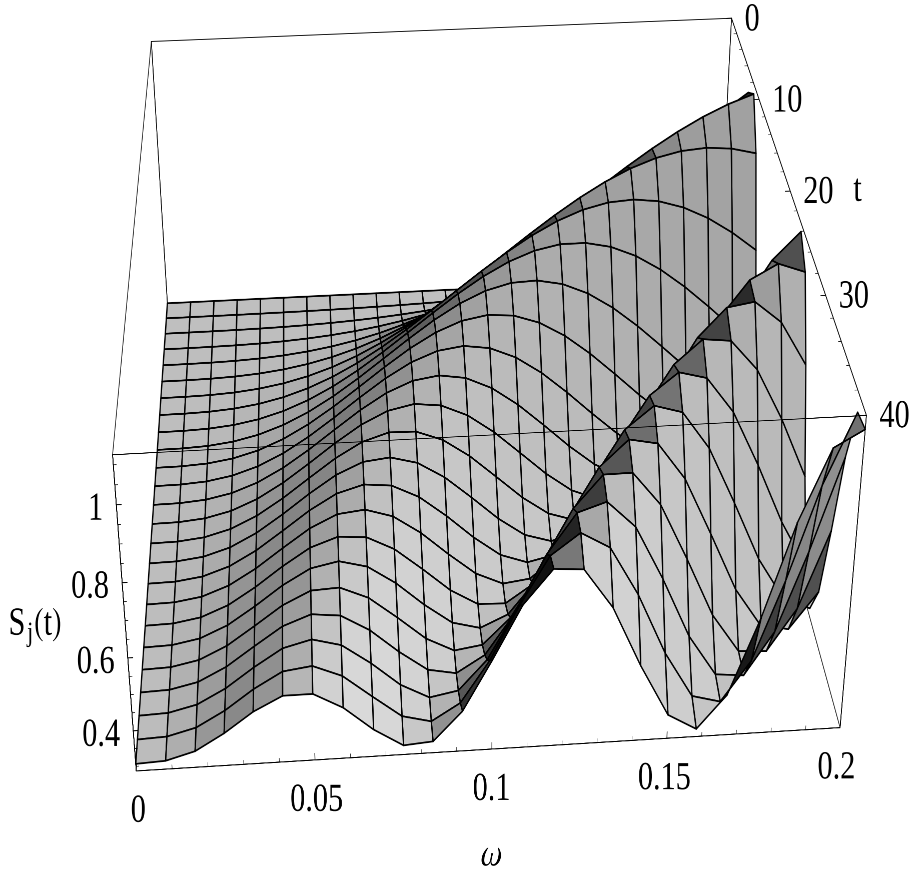

FIG. 2: The 3D graph of joint entropy of simple harmonic oscillator.

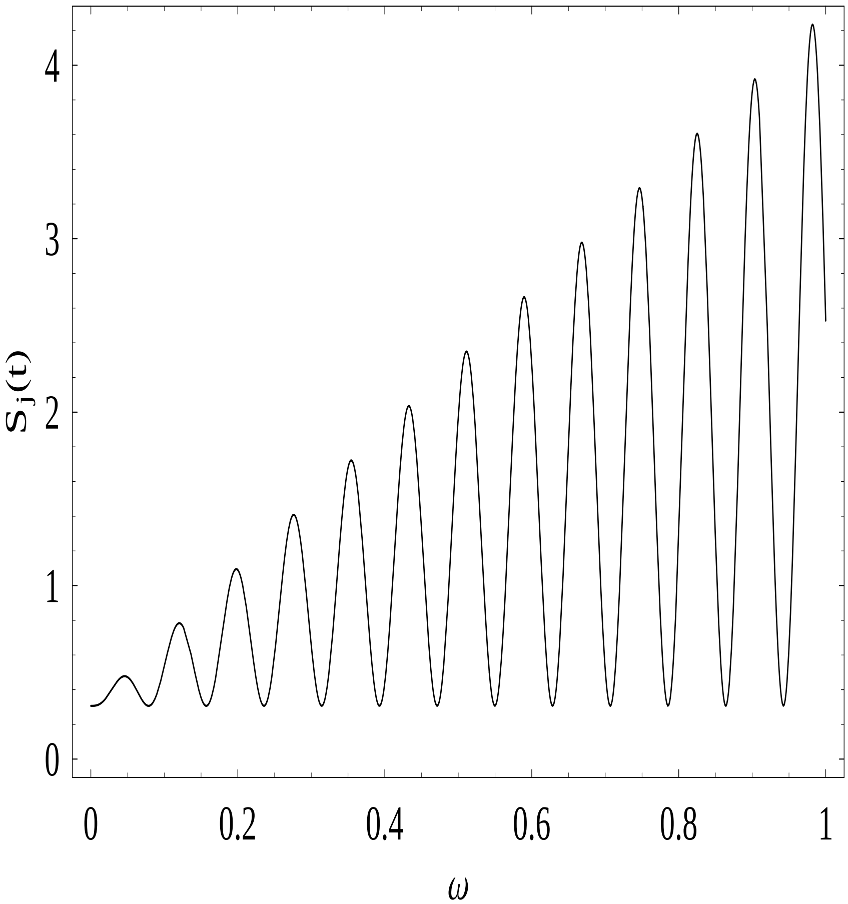

FIG. 3: The joint entropy of simple harmonic oscillator versus ω .

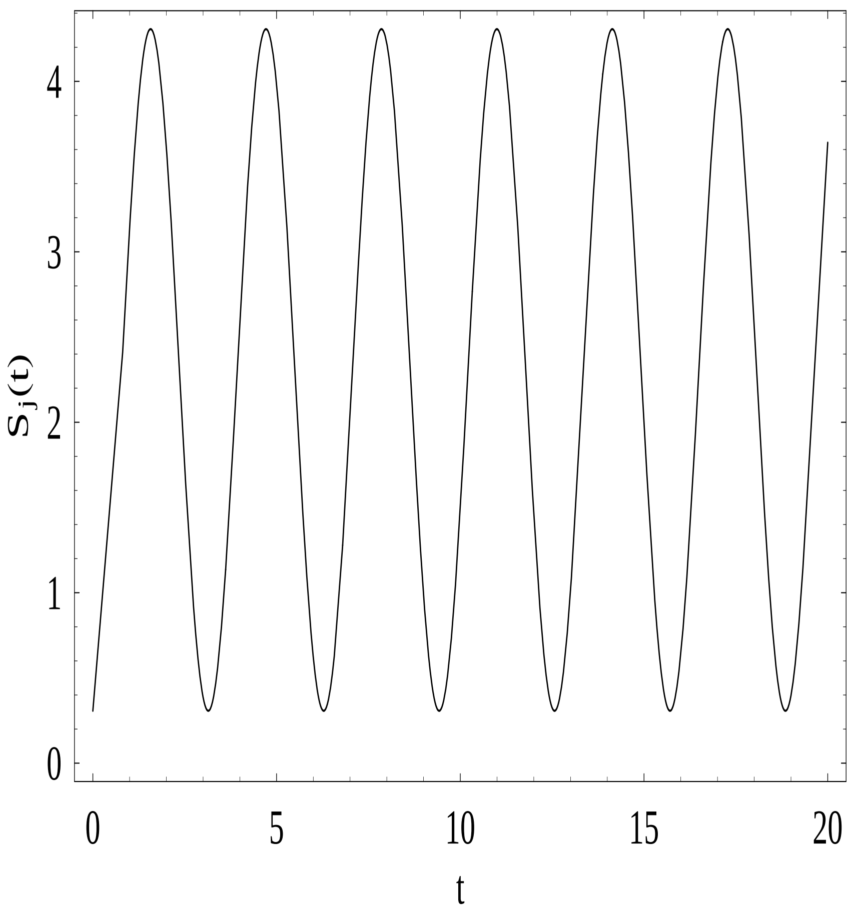

FIG. 4: The joint entropy of simple harmonic oscillator versus time.

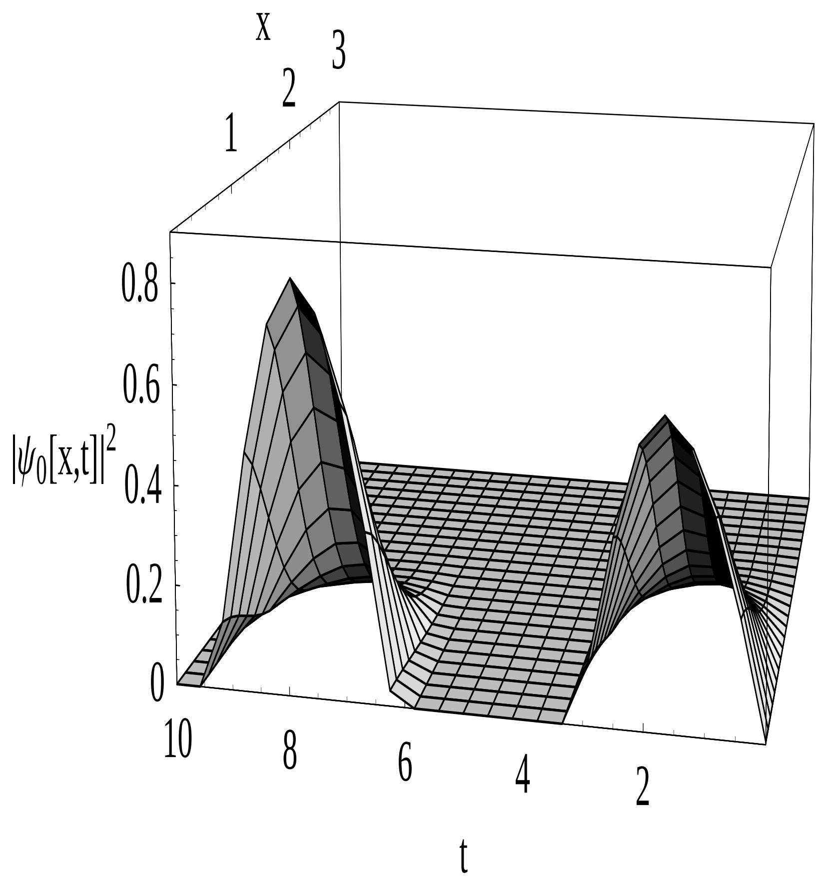

FIG. 5: The probability function as a function of time and coordinate at $\gamma = 0.1$.

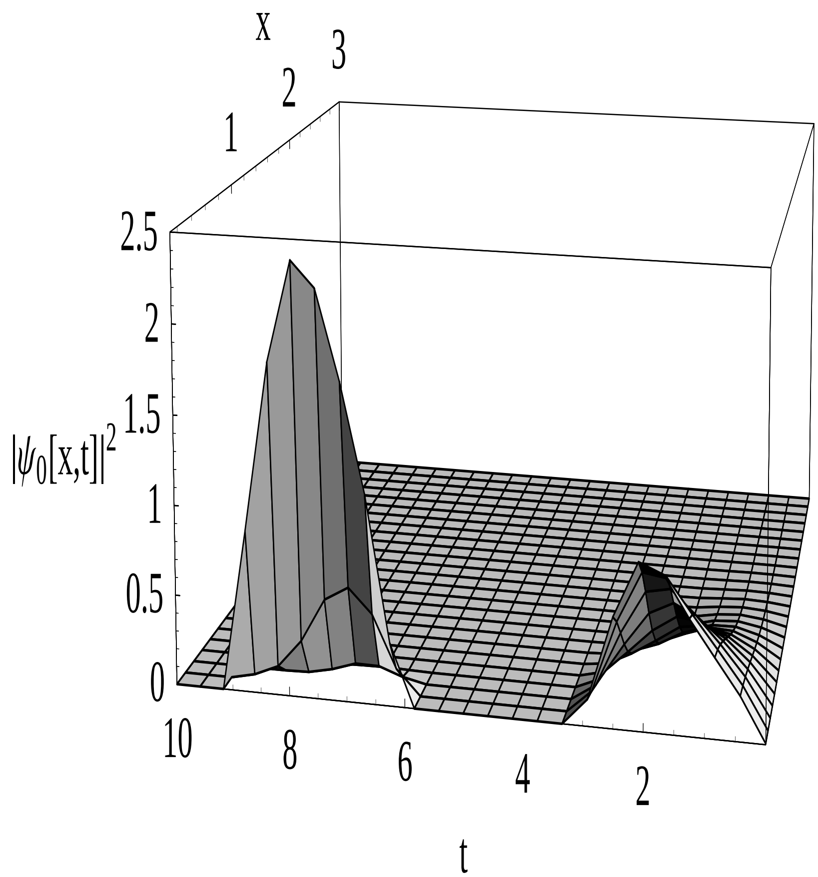

FIG. 6: The probability function as a function of time and coordinate at $\gamma = 0.5$.

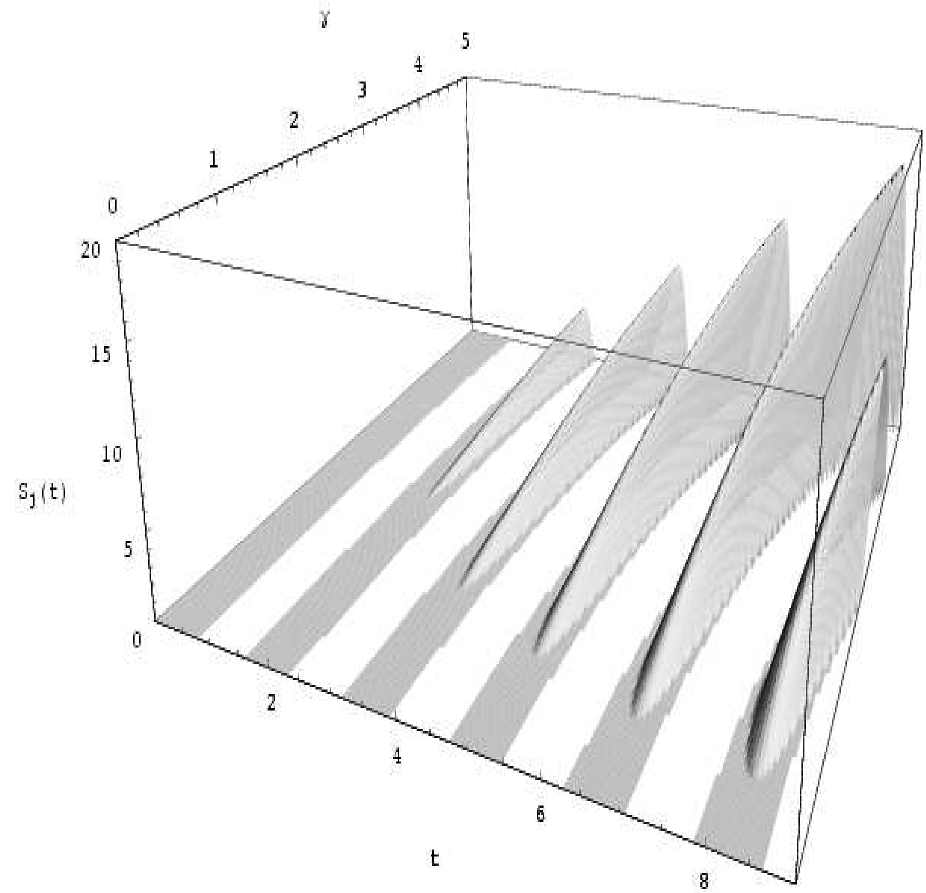

FIG. 7: The 3D graph of the joint entropy of damped harmonic oscillator for damping factor(γ) at $\omega_0 = 2$.

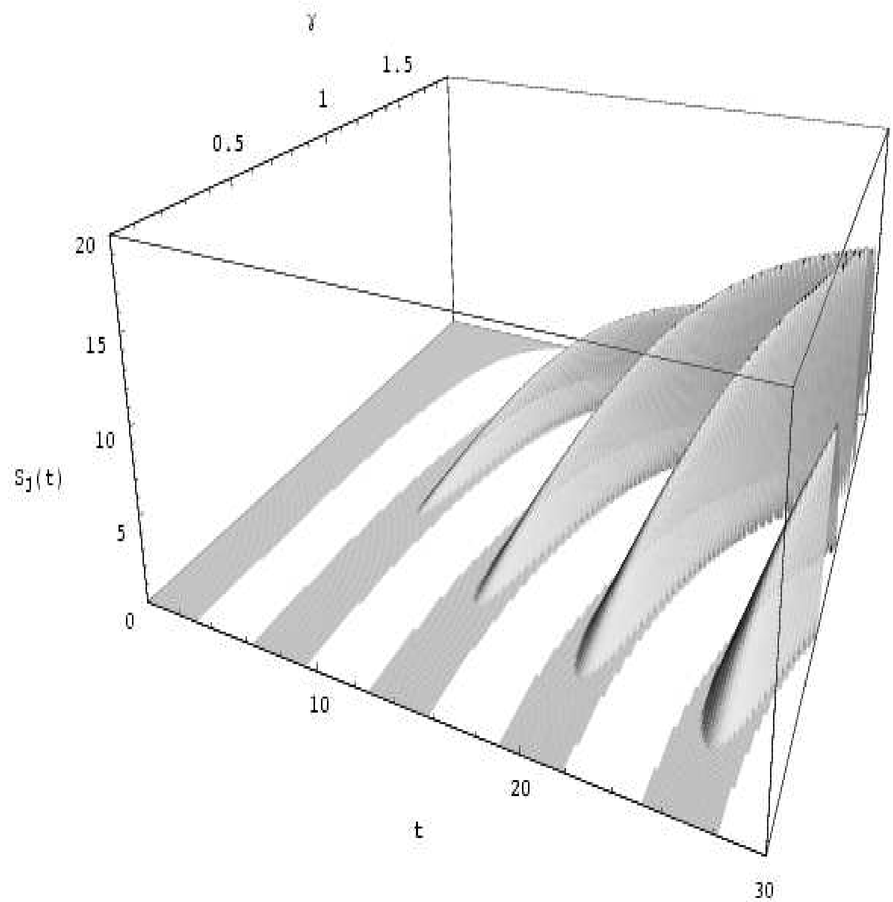

FIG. 8: The 3D graph of the joint entropy of damped harmonic oscillator for damping factor(γ) at $\omega_0 = 1$.